A Physics-Based Approach to Nonlinear Human Population Growth Modeling

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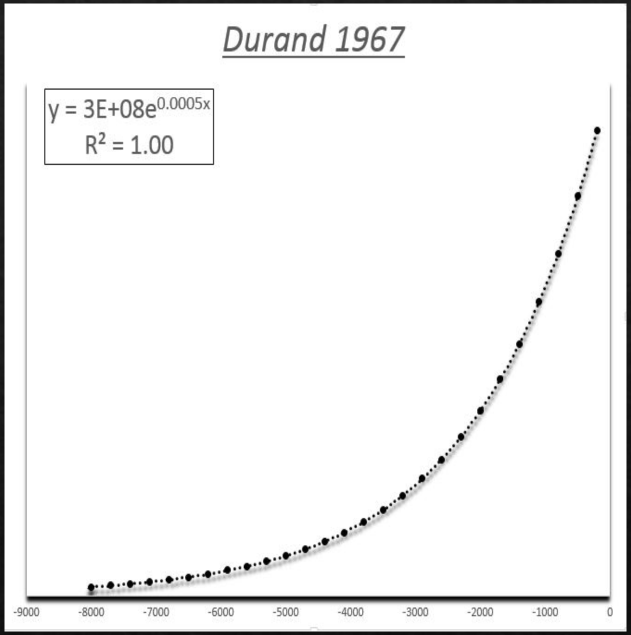
Data Compilation

- Using available census data from the USCB, UN, and other sources, a "total" was established and deemed the "canonical" dataset.
- This canonical set was compared to known historical events that significantly reduced the population:
  - Antonine Plague (165 – 180 AD)
  - Plague of Justinian (500 – 700 AD)
  - The Bubonic Plague (1350 AD)
  - Black Death (1350 AD)
  - Plague of Justinian (541-542 AD)
  - WWI (1914-1918 AD)
  - WWII (1939-1945 AD)

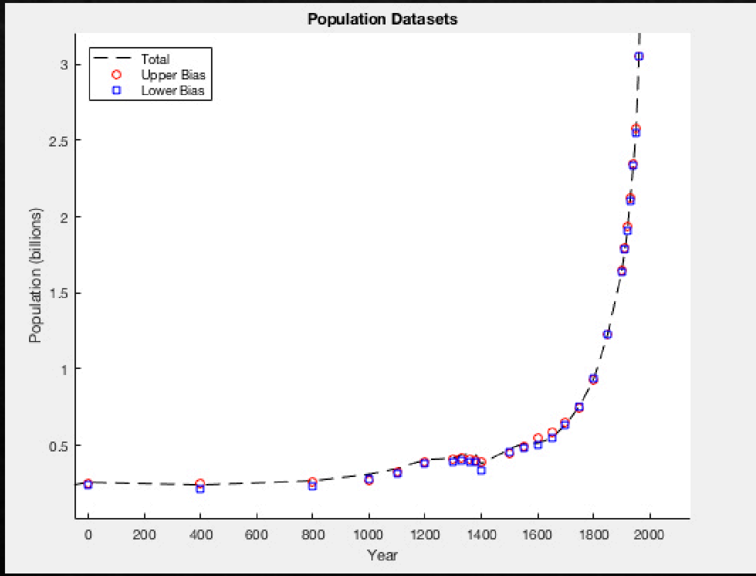
Creating A Population Data Bias

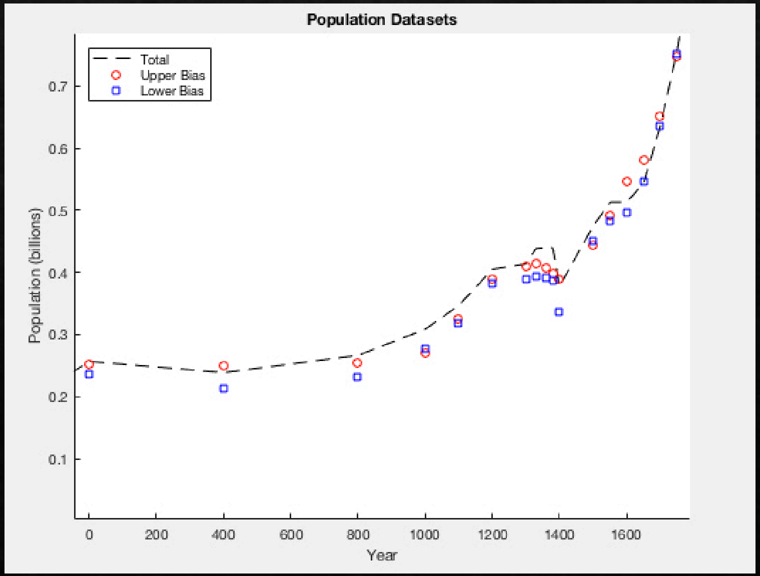
- The main issue with creating a bias was finding realistic data.
- The Durand 1967 data was a perfect exponential fit, which is impossible.



Formation of the Upper and Lower Bias

- From the comparison of canonical data to the aforementioned events, some data sets were eliminated accordingly and the remaining were deemed the “bias”.





Models

- In order to derive a functional model of the human population over time, the Law of Mass Action and Chemical Kinetics are used to develop a relationship from known models:
  - Power
  - Logistic
  - Exponential

Power Model

- Considering the growth of population as a function of the interaction of its members leads to proportionality of the population growth to the square of the population:

$$\frac{dN}{dt} = aN^2$$

- Which has the solution:

$$\mathrm{N}(t) = \frac{N_0}{(t_o - t)^a}$$

Exponential Model

- Another known model that represents the population growth as proportional to an exponential function:

$$N(t) = N_0 e^{a(t-t_0)}$$

Logistic Model

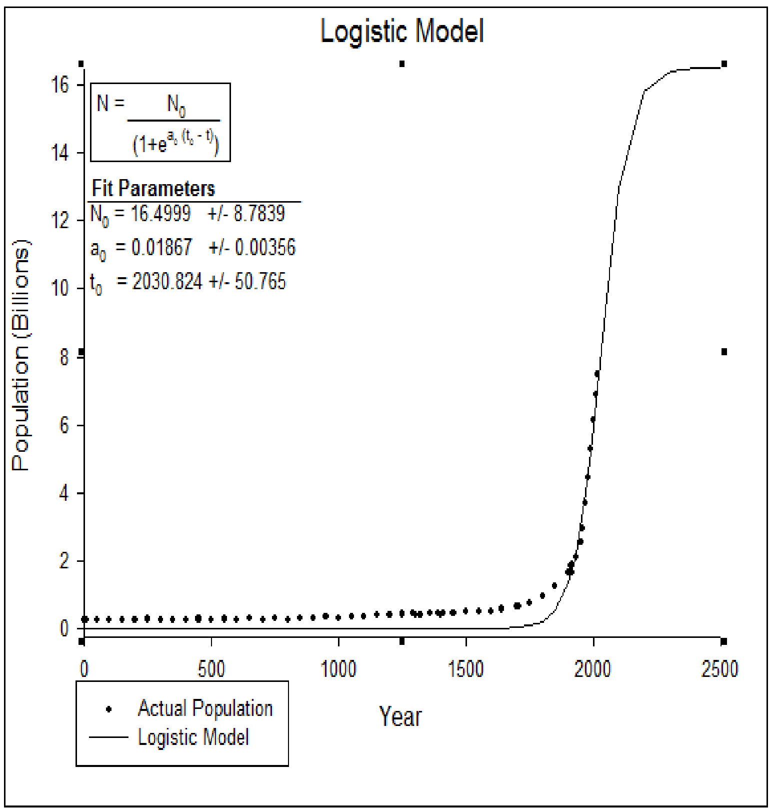
- Developed by Lotka and Voltera, the Logistic model represents the populations growth rate as proportional to the population, but assuming that the growth rate is also a function of the carrying capacity:

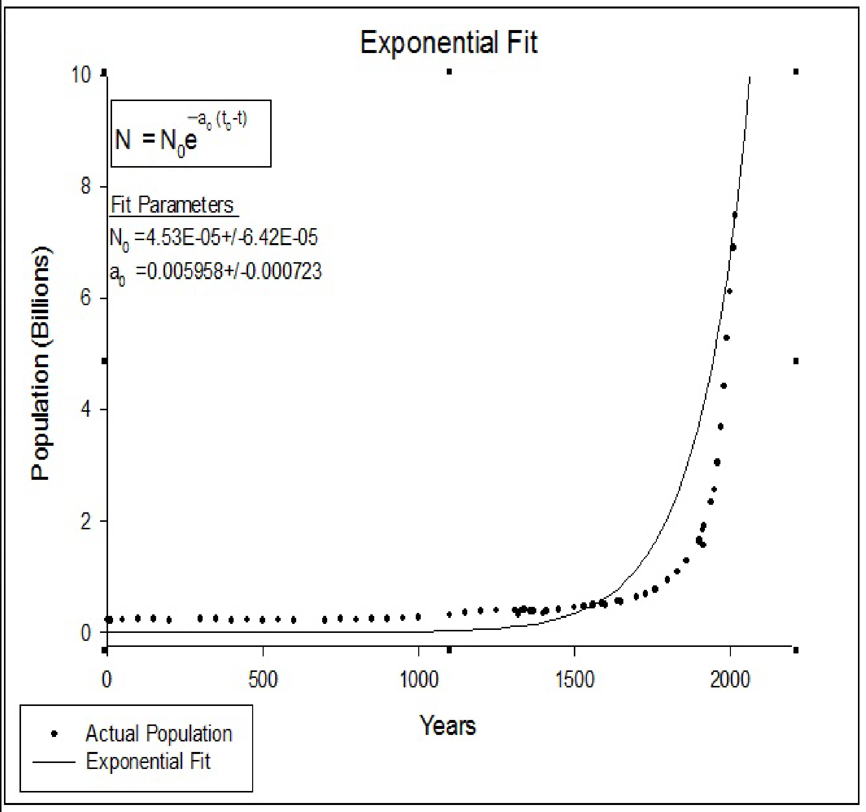
$$\frac{dN}{dt} = a\left(1 - \frac{N}{N_0}\right)N$$

- Which has the solution:

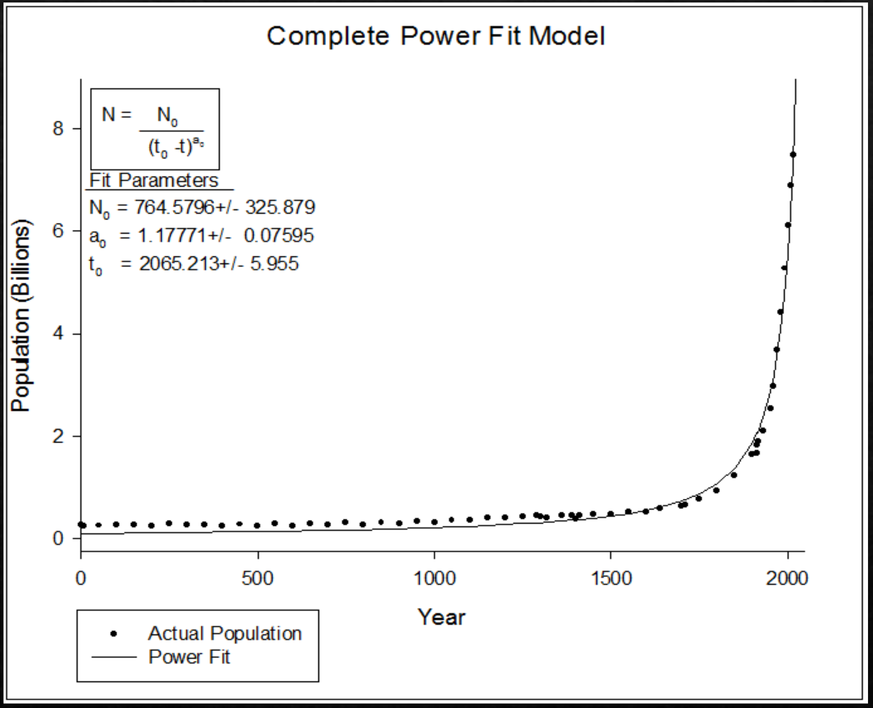
$$N(t) = \frac{N_0}{1 + \mathrm{e}^{-a(t-t_0)}}$$

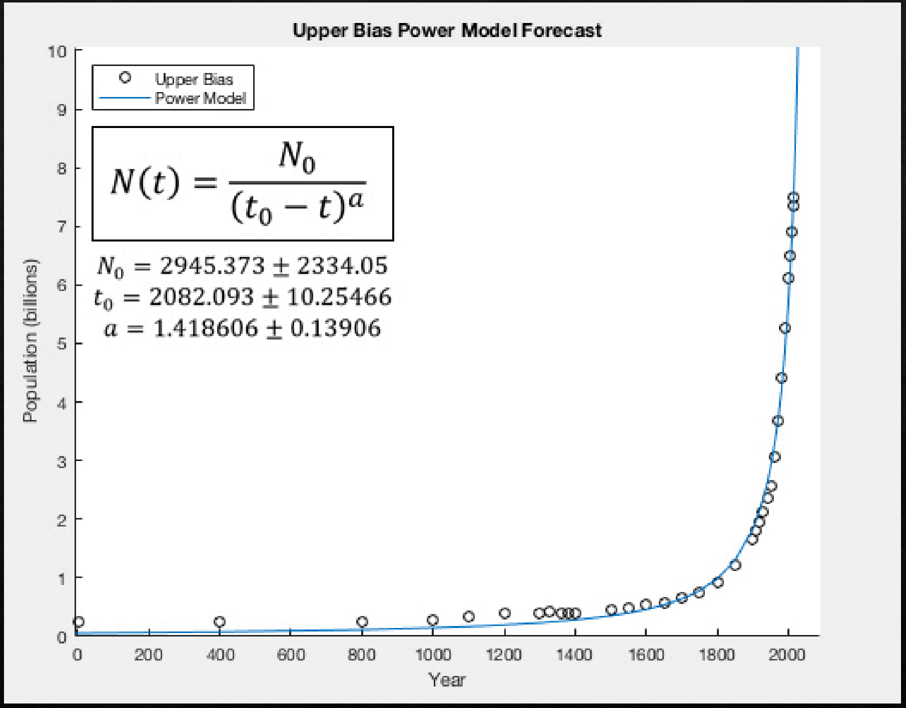
Exponential Vs. Logistic





Power Model Comparison

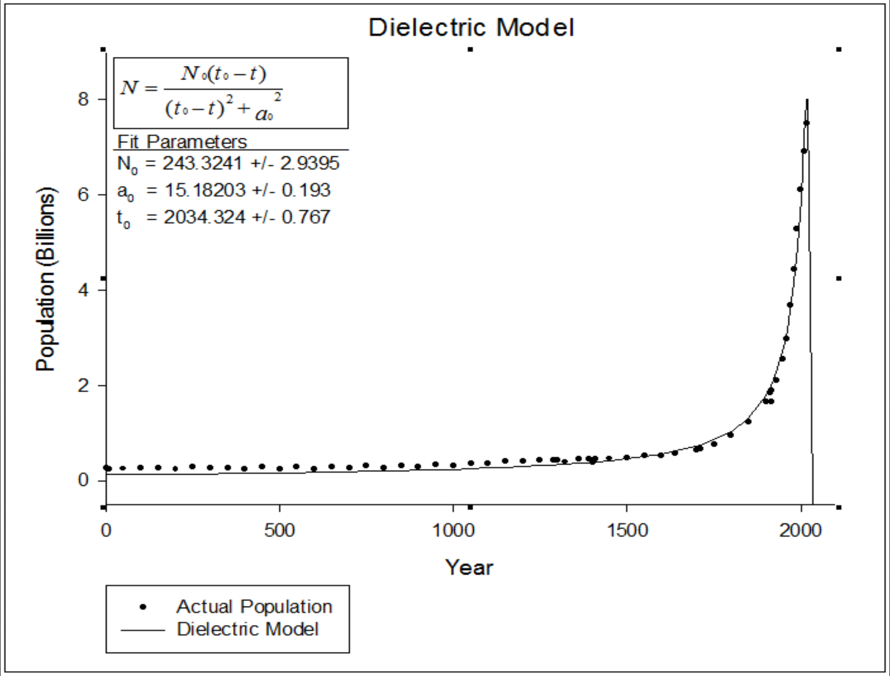


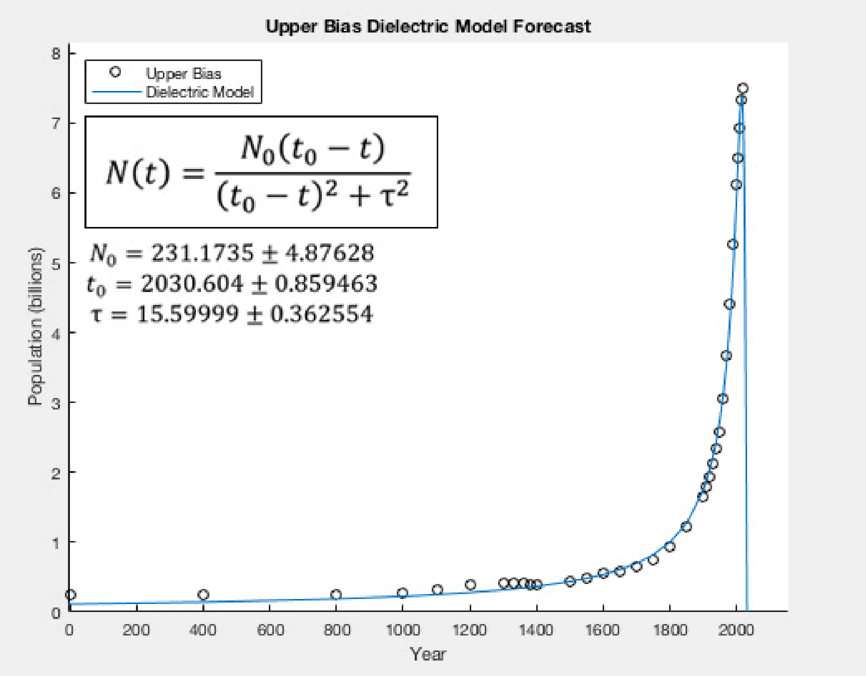


Transition/Extinction Models

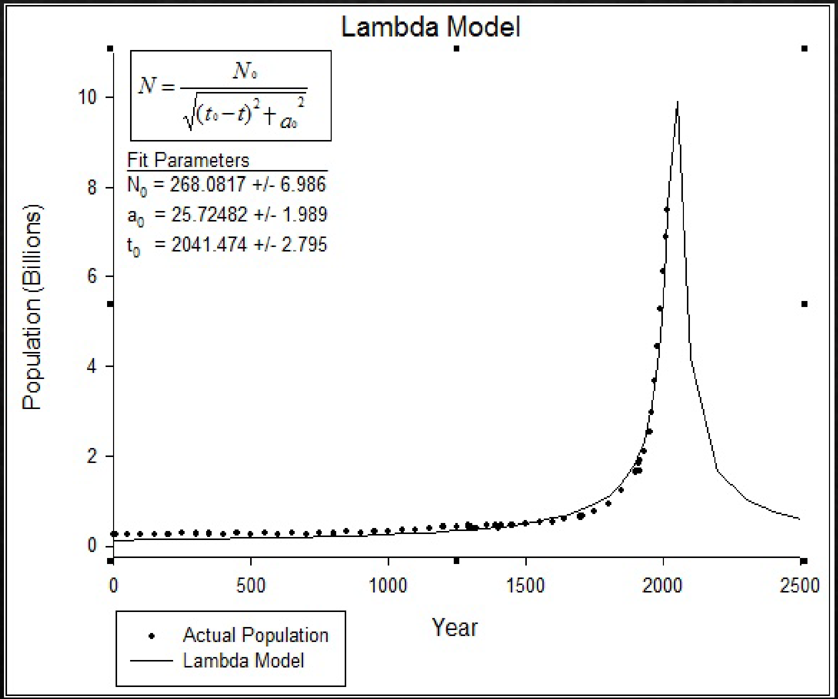
- The Power, Logistic, and Exponential models of the population were then compared to other models:
  - Stabilization
  - Lambda
  - Dielectric
  - Extinction

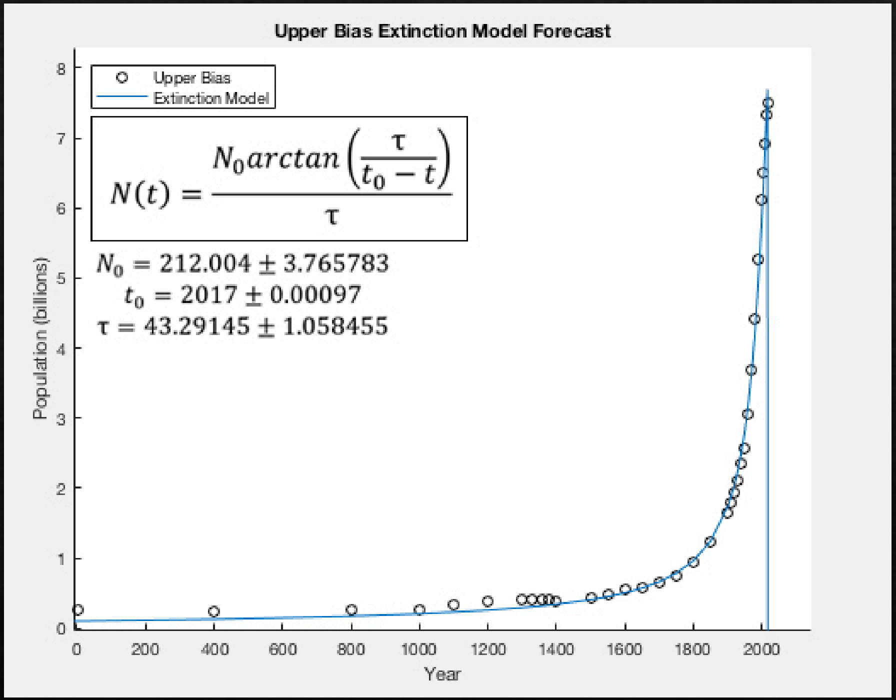
Dielectric Model Forecast



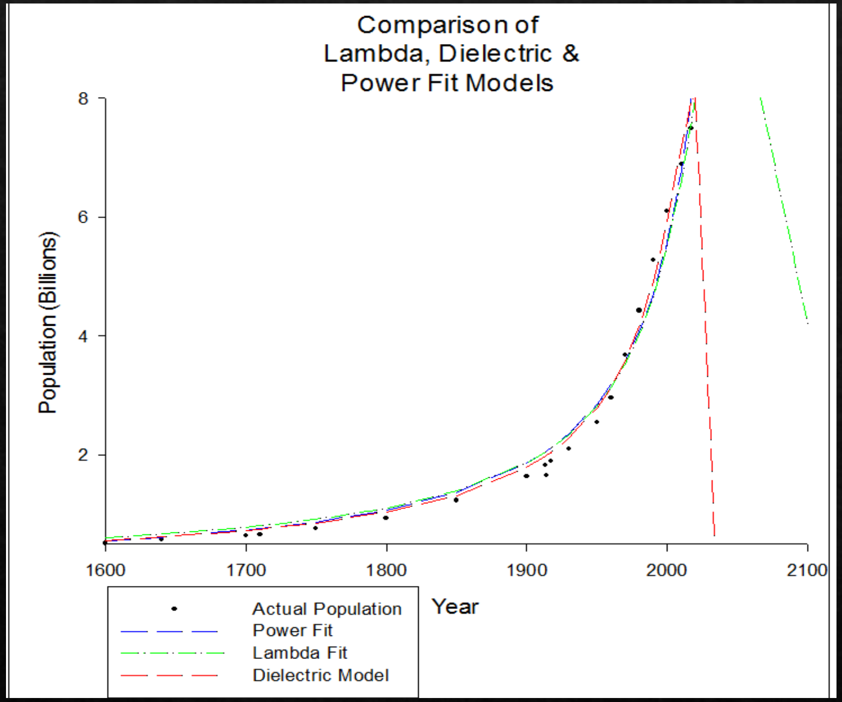


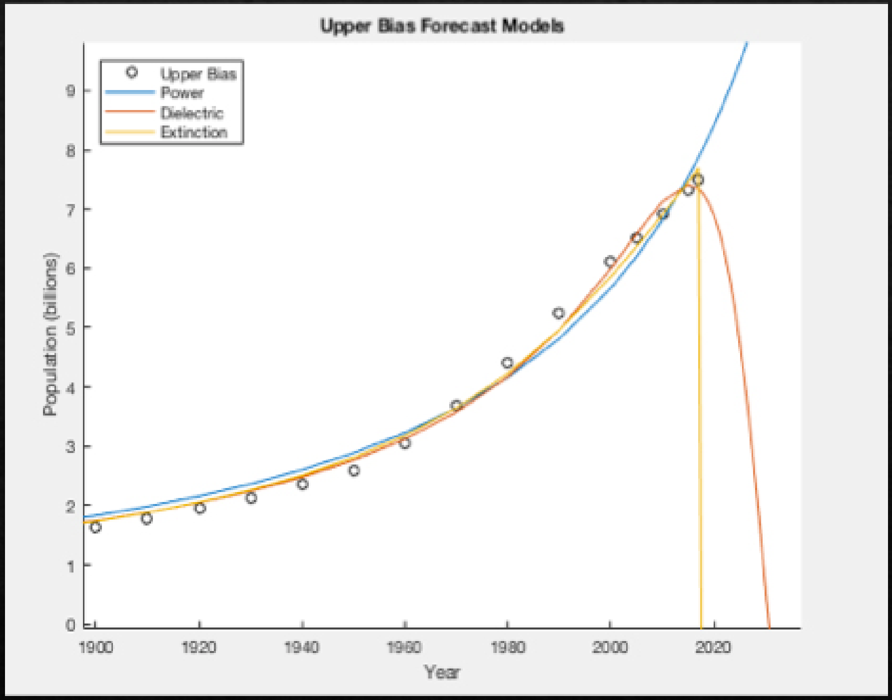
Lambda & Extinction Model Forecast





Comparison of the Three Best Models





Parameter Optimization

- Using Excel's Solver tool, the parameters of each model were optimized by a minimization routine on the sum of squares.
- Error estimates for each parameter were also calculated.

Model Comparison

- After completing the optimization for both a full data set (408 entries) and a recapitulated data set (41 entries), the sum of squares and R-squared values were compared for each of the models.

Model Optimization

Curve Fit Parameters		Full Data Set (408 data points)						Recapitulated Data Set (41 data points)			
		Total	Error	Prather	Error	Fickess	Error	Total	Error	Prather	Error
Power	R-squared	0.991569462		0.99321966		0.991009		0.99274689		0.99354517	
	SoS	14.08633805	0.18626705	11.3643459	0.16730511	12.9942	0.0204	1.6547329	0.20598313	1.475406	0.19450173
	tt	2078.5528	3.76242225	2078.74608	3.39101869	2065.213	2.9348	2081.73174	10.8243488	2082.09272	10.2546555
	n	2945.38	874.546819	2945.38332	785.87004	764.5796	195.02	2945.37436	2478.89855	2945.37262	2334.04985
	a/tau	1.4286	0.05243381	1.42799547	0.04709256	1.17771	0.0412	1.42044767	0.14784048	1.41860558	0.13906023
Exponential	R-squared	0.988740616		0.98964406		0.990326		0.9902565		0.99089983	
	SoS	31.35530193	0.27790251	28.6841182	0.26547495	26.92651	0.2608	4.28591421	0.3315045	4.09108245	0.323882
	tt	2017.0001	0	2017.00001	0	2036.928	30000000	2017.00001	125847418	2017.00001	86941412.4
	n	7.569	0.07100809	7.53796842	0.04522914	10.14552	4000000	7.51991416	13308977.6	7.51328611	9100564.05
	a/tau	0.01439	0.00022816	0.01425657	0.00021717	0.014592	0.0002	0.01406331	0.00069425	0.01393194	0.00067232
Logistic	R-squared	0.984808631		0.98541953		0.990316		0.98691214		0.98739177	
	SoS	39.25854344	0.31095968	37.7022834	0.30473393	26.97281	0.2578	5.62795909	0.37987714	5.46175233	0.37422578
	tt	2017.00001	12.9032181	2017.00001	12.49441	2390.181	1801.6	2017.00001	38.2891048	2017.00001	38.1276961
	n	14.402	2.05414874	14.6112422	2.01363854	1782.023	46080	14.6250364	6.05994199	14.6112422	5.97274806
	a/tau	0.02046	0.00124174	0.02041795	0.00119816	0.014592	0.0009	0.02060586	0.00395506	0.02041795	0.00387405
Stabilization	R-squared	0.988192723		0.98977502		0.991716		0.9889177		0.98967043	
	SoS	15.90055865	0.19765554	13.8308845	0.18434348	14.27799	0.1124	2.40459694	0.24830693	2.26082106	0.24076912
	tt	2031.006	0.62813244	2037.76936	0.58738935	2041.474	2.795	2020.2862	47131770.4	2020.58547	45700999.4
	n	286.399	3.47520515	286.661818	3.24671027	268.0817	6.986	295.082383	12.446001	297.091591	12.1374382
	a/tau	19.74		13.0446049		25.72482	1.989	33.4570715	47131770.4	33.4109031	45700999.4
Lambda	R-squared	0.988192726		0.99857511		0.991716		0.98891763		0.98967165	
	SoS	15.90054997	0.19789875	2.91904745	0.08479252	14.27799	0.1124	2.40459707	0.24830694	2.26081978	0.24076905
	tt	2050.747	3.28174635	2018.24738	0.43127403	2041.474	2.795	2053.74285	36.5039594	2054.00332	174.9286
	n	286.383	7.67169526	202.903708	1.59851981	268.0817	6.986	295.079701	14.4472945	297.135358	64.5826243
	a/tau	0.00399	22842.9221	27.718717	0.17775273	25.72482	1.989	2.9572E-05	431077.277	1.0032E-05	691493.758
Dielectric	R-squared	0.996344272		0.99747235		0.995282		0.99734214		0.99789509	
	SoS	5.569677519	0.11712565	3.82574699	0.09707225	6.555228	0.1296	0.57614326	0.12154384	0.45639826	0.10817818
	tt	2028.941151	0.36858809	2029.12412	0.31188995	203.324	0.767	2030.43858	0.95203756	2030.6035	0.8594632
	n	226.8775394	1.85844749	227.64651	1.55681382	243.3241	2.9395	229.374424	5.42799116	231.173462	4.87627993
	a/tau	15.5344514	0.14888181	15.5893409	0.12371873	15.18203	0.193	15.4797372	0.40519674	15.5999877	0.36255406
Extinction	R-squared	0.996009256		0.99717717		X		0.99701889		0.99759394	
	SoS	6.336324678	0.12492682	4.48750319	0.10513308	X	X	0.66687075	0.1307641	0.53994521	0.11766371
	tt	2017.00001	0.00086562	2017.00001	0.00072962	X	X	2017.00001	0.00107859	2017.00001	0.00096998
	n	209.5600344	1.32880797	209.926331	1.12144754	X	X	210.485436	4.16068434	212.004032	3.76578283
	a/tau	41.89431019	0.41350618	42.0401598	0.34947166	X	X	42.9652131	1.16943191	43.2914534	1.05845462

Optimization Conclusions

- Generally, the R-squared value improved for the recapitulated data.
- The Power model was a better fit than both the Logistic and Exponential models.
- The Dielectric and Extinction models had consistently high R-squared values for both data sets.

Forecasting Conclusions

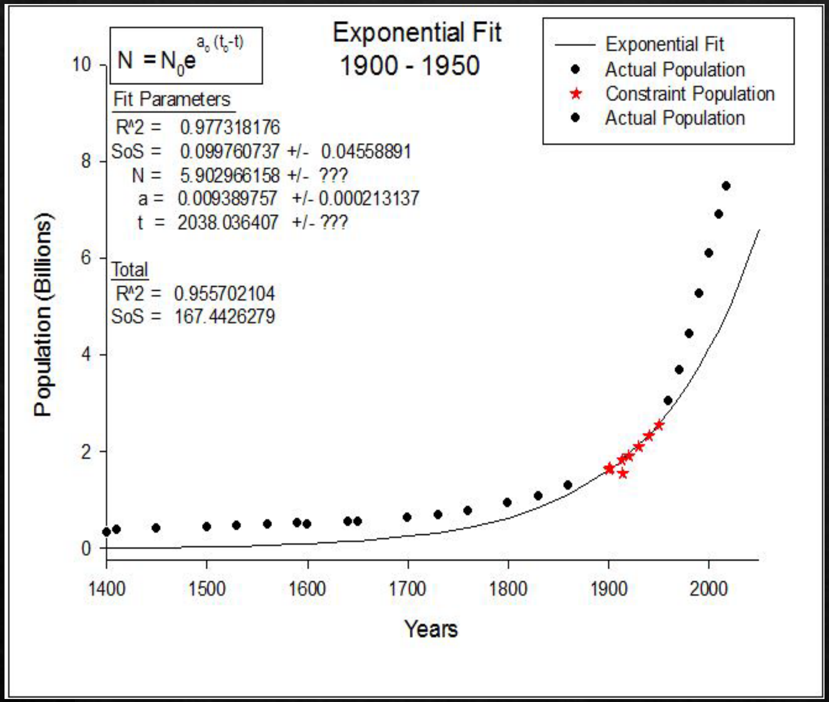
- The power model has a large error associated with one of its parameters, and although having a high R-squared value, it is unlikely that the population will continue increasing indefinitely.
- The Extinction model reaches a critical point at the last known input (2017) thus is not likely an accurate prediction.
- The Dielectric model remains the most likely description of human population growth.

Timeframe Extrapolation

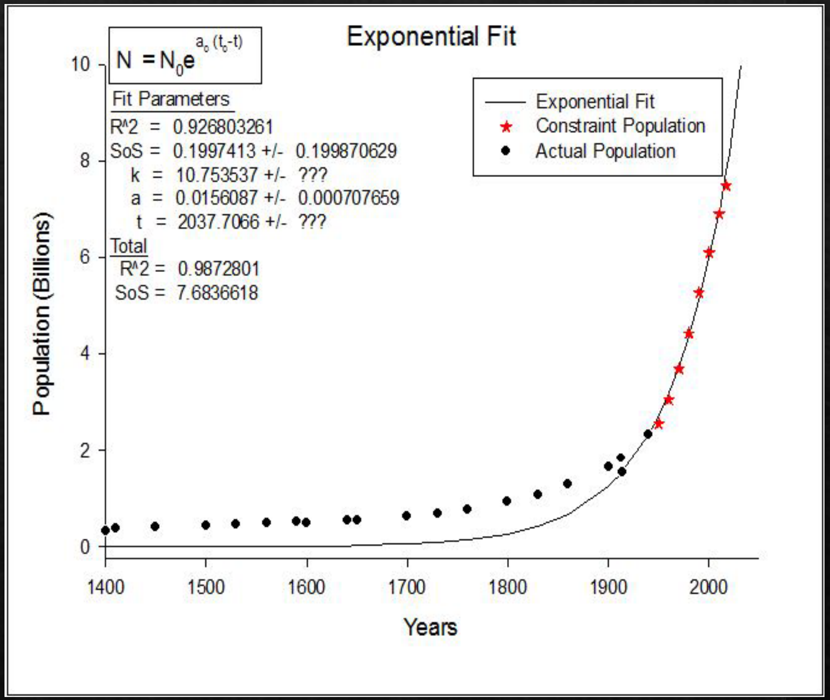
- So if parameter optimization works for the total timeframe does it work well looking at a differential time segments.
  - Therefore, to test this take only data from time A-B and do the optimization processes done in the prior slides.
  - So does this work well?

Exponential Model from 1900 – 1950 Vs. 1950 – Present Constraint Models

1900 - 1950

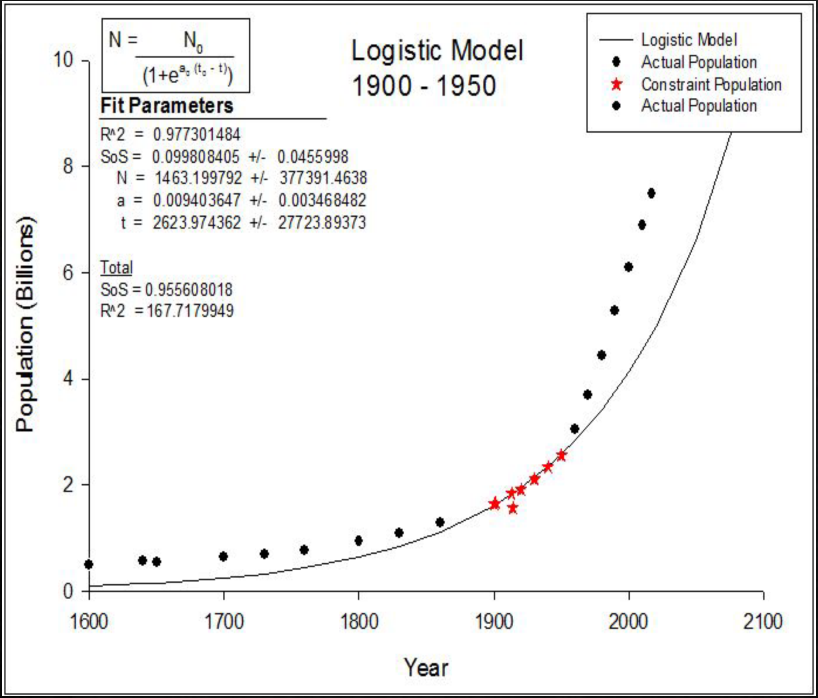


1950 - Present

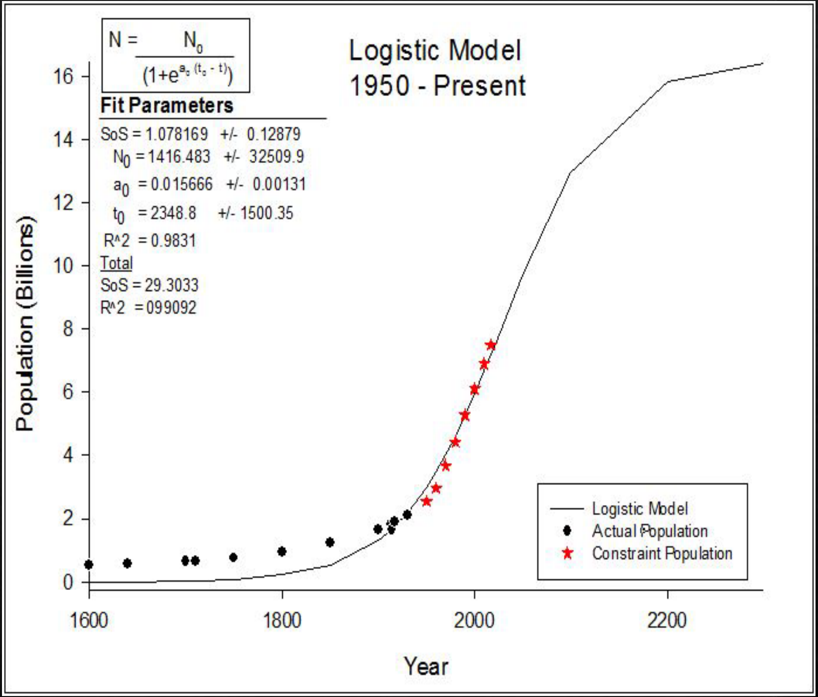


Logistic Model from 1900 – 1950 Vs. 1950 – Present Constraint Models

1900 - 1950

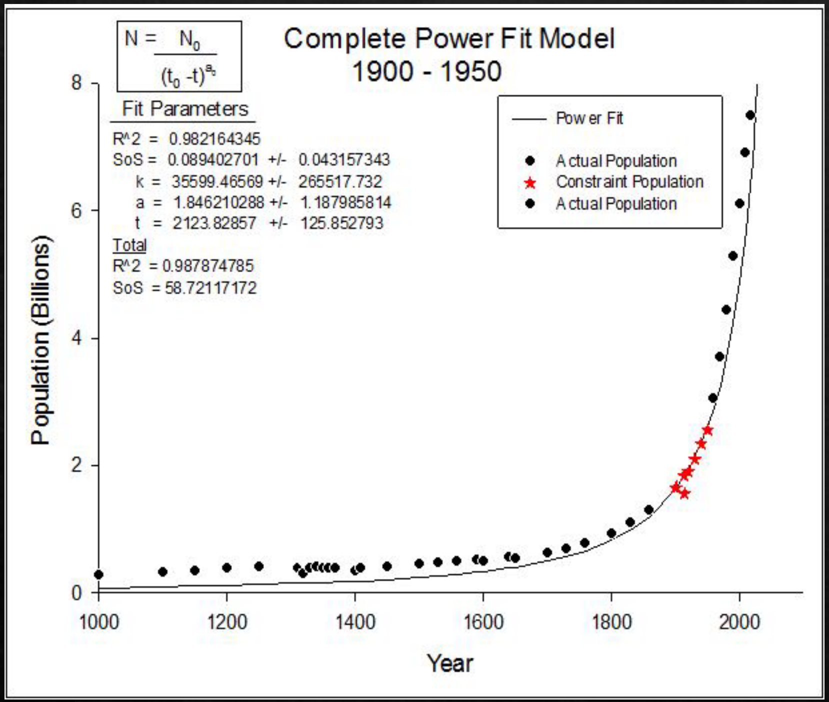


1950 - Present

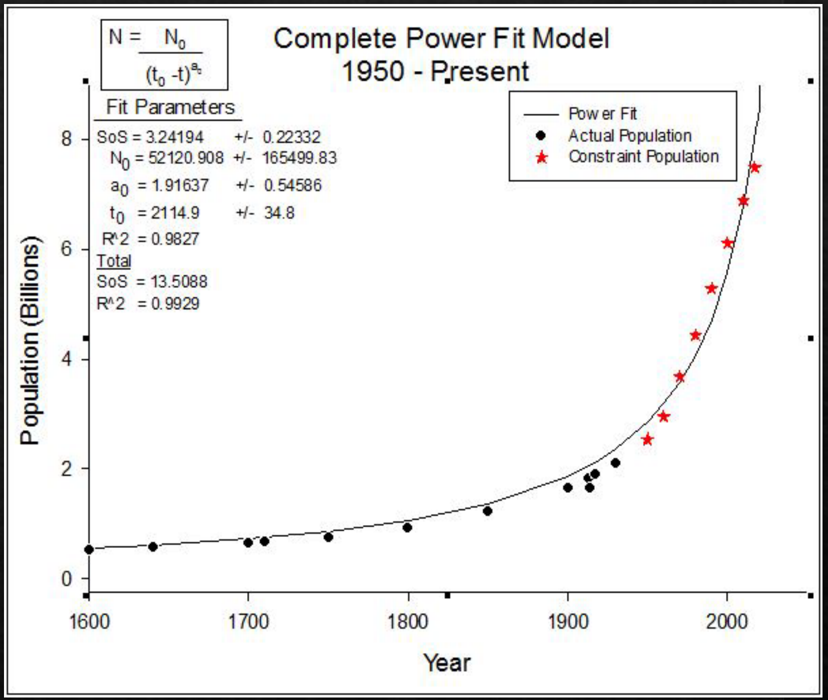


Power Model from 1900 – 1950 Vs. 1950 – Present Constraint Models

1900 - 1950

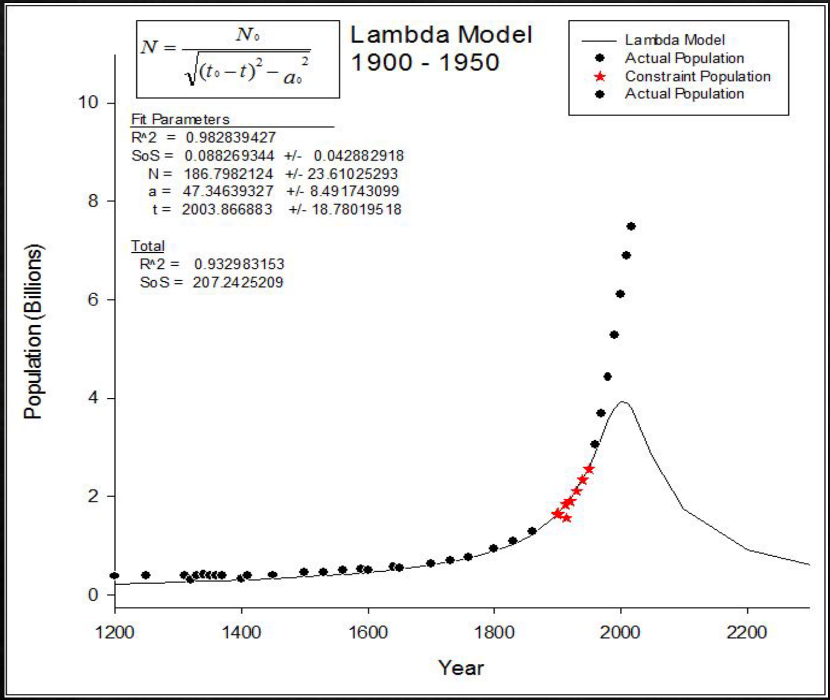


1950 - Present

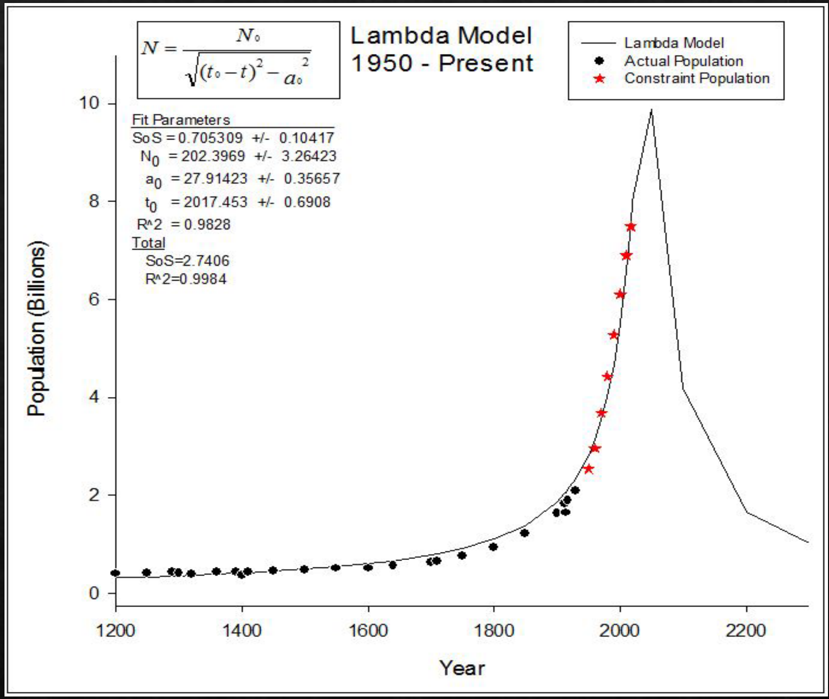


Lambda Model from 1900 – 1950 Vs. 1950 – Present Constraint Models

1900 - 1950

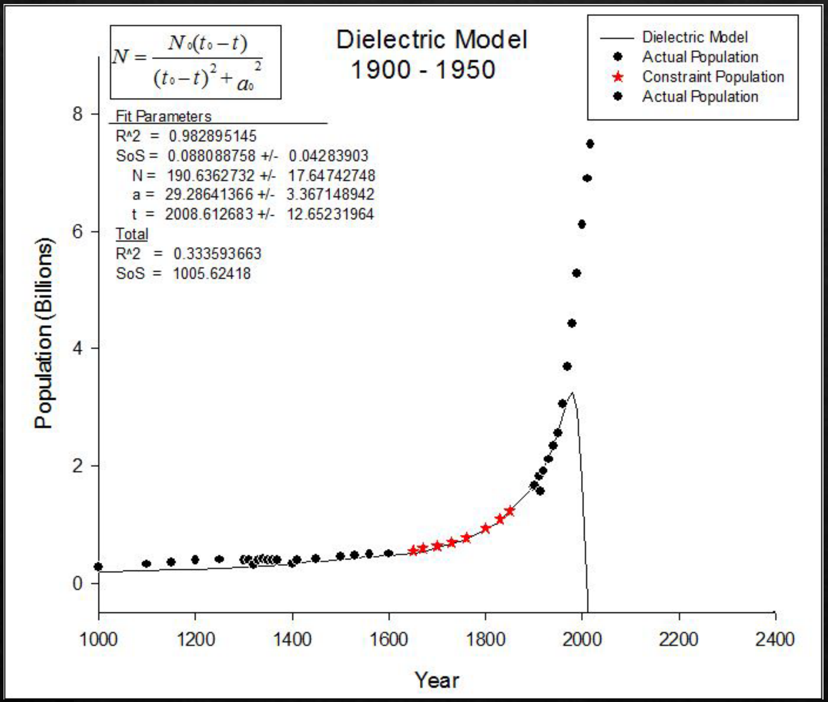


1950 - Present

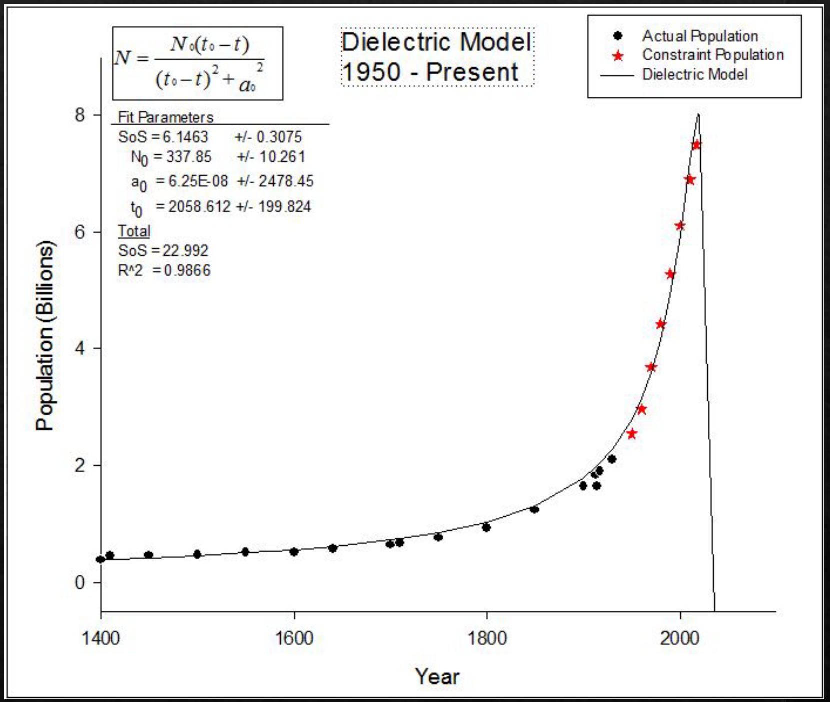


Dielectric Model from 1900 – 1950 Vs. 1950 – Present Constraint Models

1900 - 1950

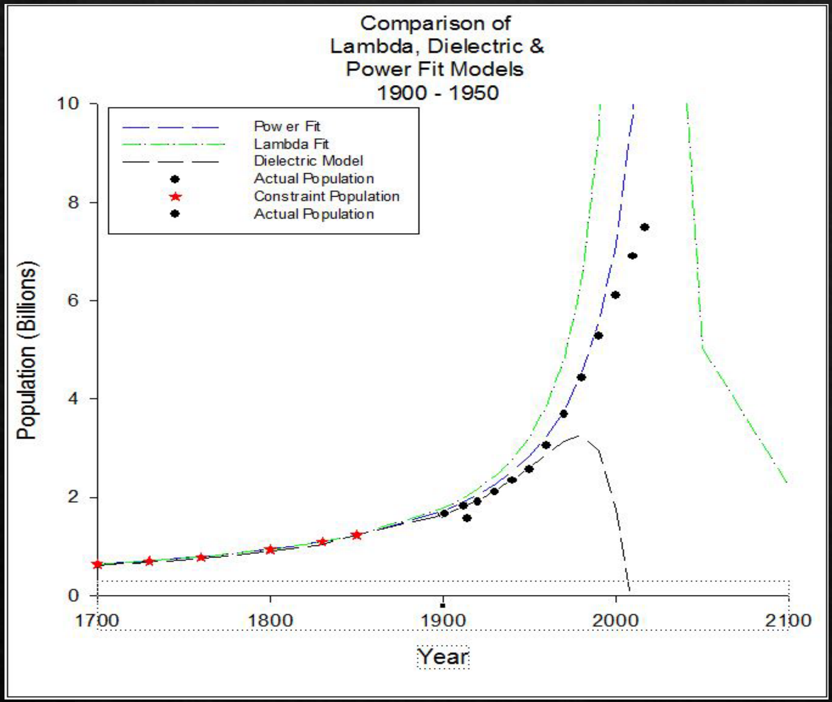


1950 - Present

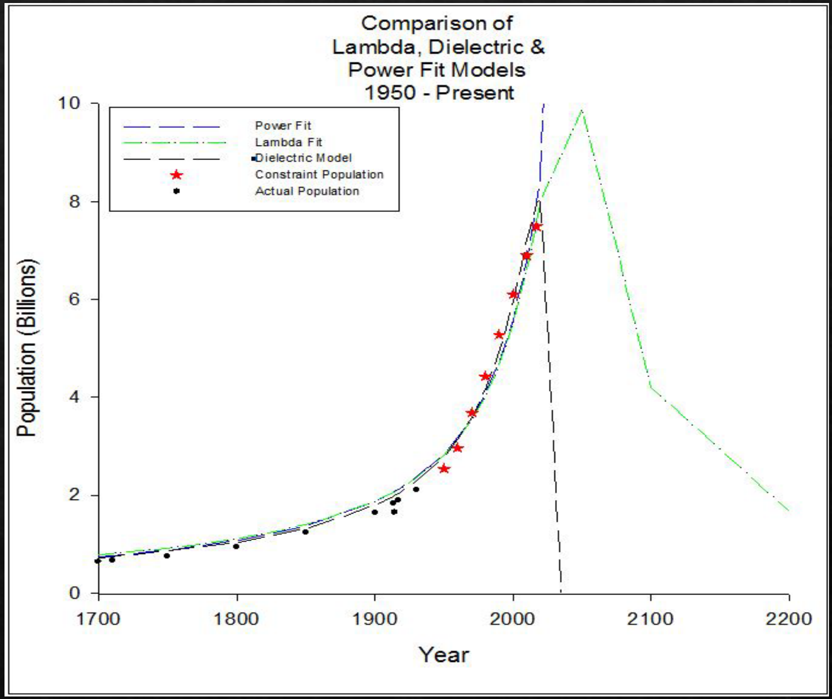


Comparison of the 3 best models from 1900 – 1950 Vs. 1950 – Present Constraint Models

1900 - 1950



1950 - Present

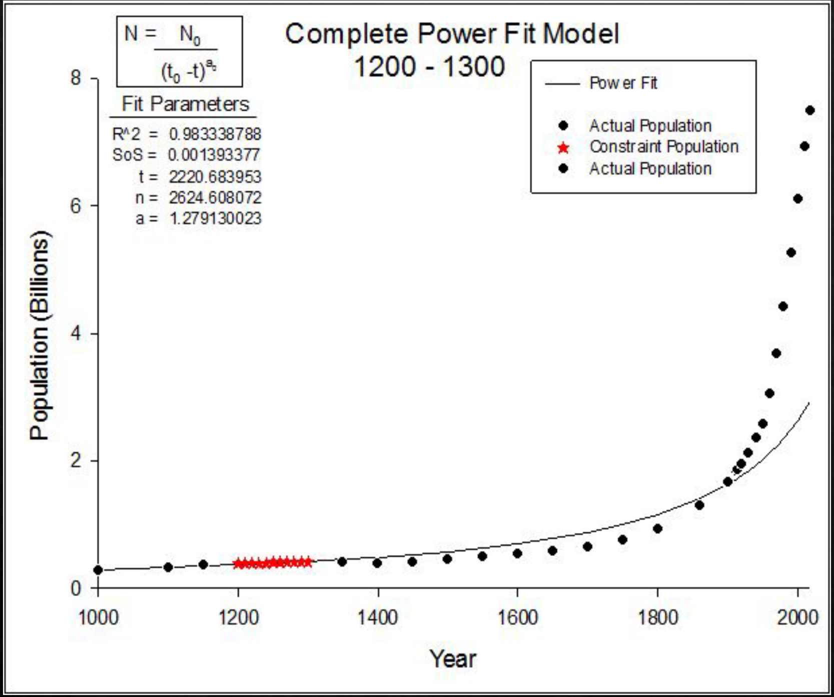


Trend? (Or Lack There Of?)

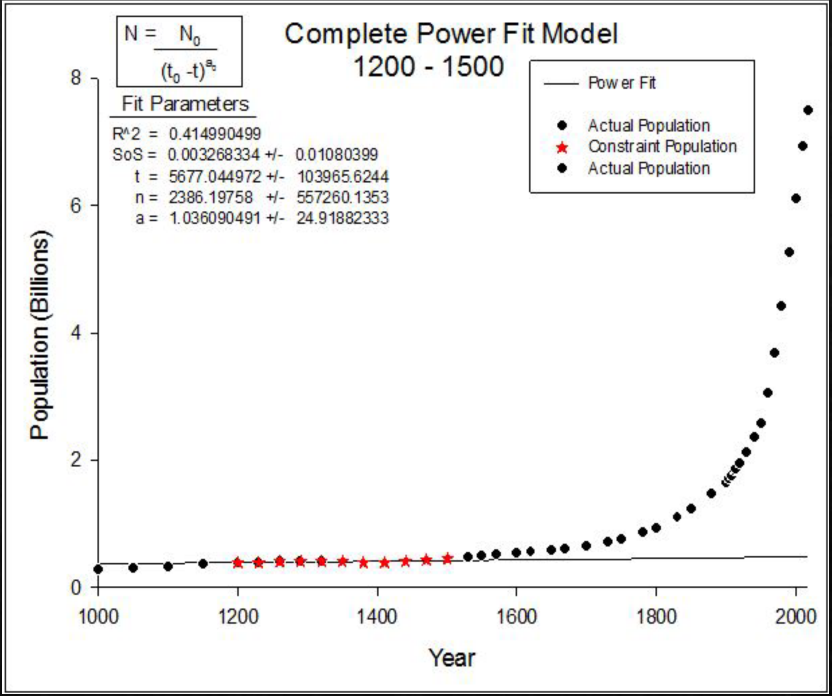
- From the previous slides we saw that there is definitely a downward trend in the optimization for the 1950 – Present Vs. 1900 – 1950 curves.
- Does this trend continue to decline if we got to lower dates?
- To test this we will look at the 1200 -1300 and 1200 - 1500 timeframe optimization.

Power Model from 1200 - 1300 Vs. 1200 – 1500 Constraint Models

1200 - 1300

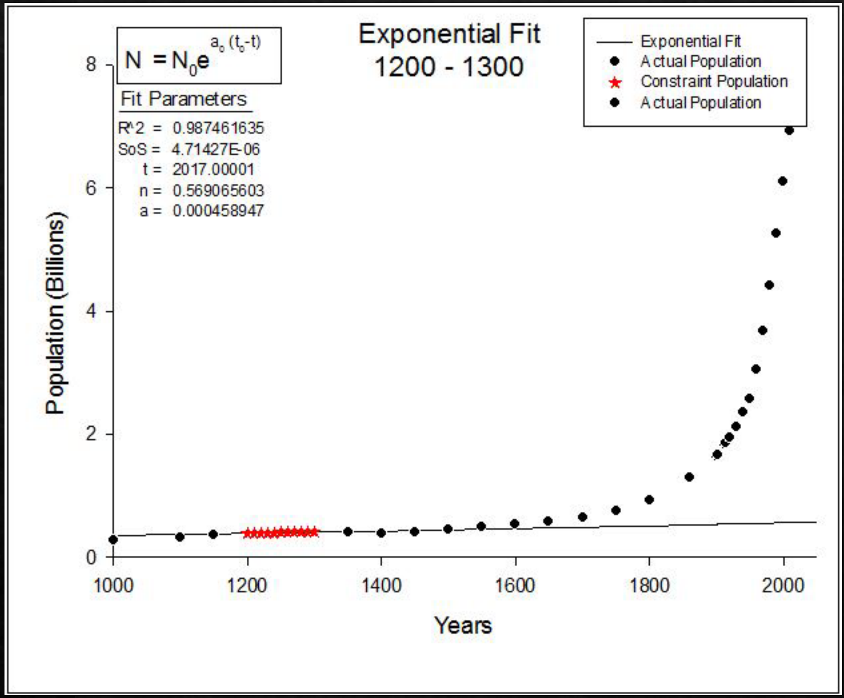


1200 - 1500

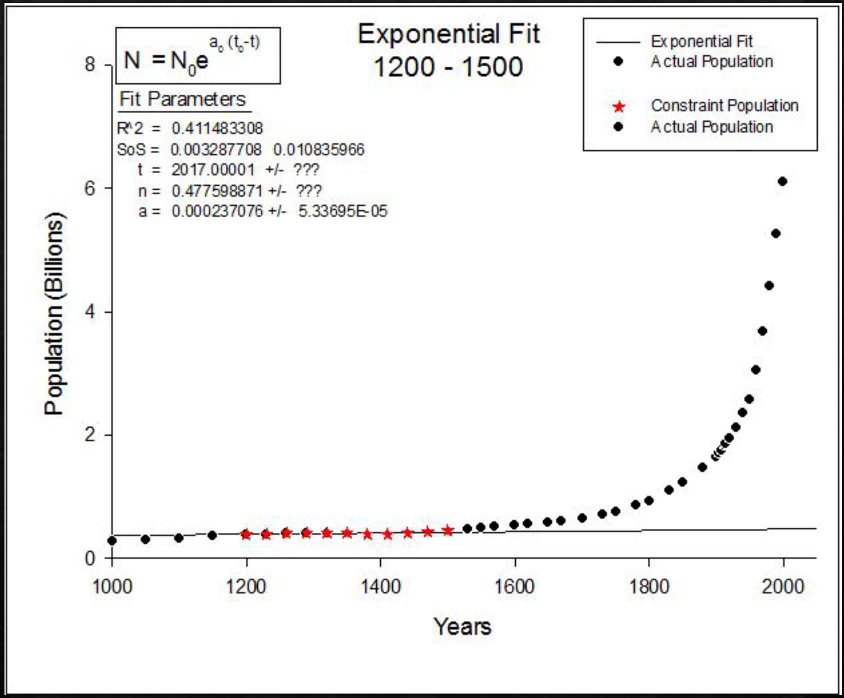


Exponential Model from 1200 – 1300 Vs. 1200 – 1500 Constraint Models

1200 - 1300

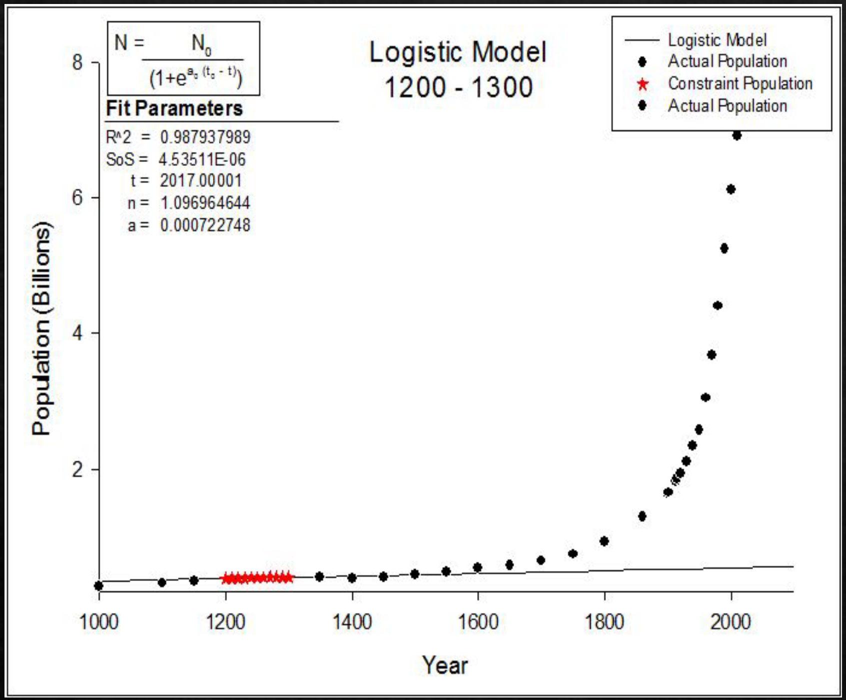


1200 - 1500

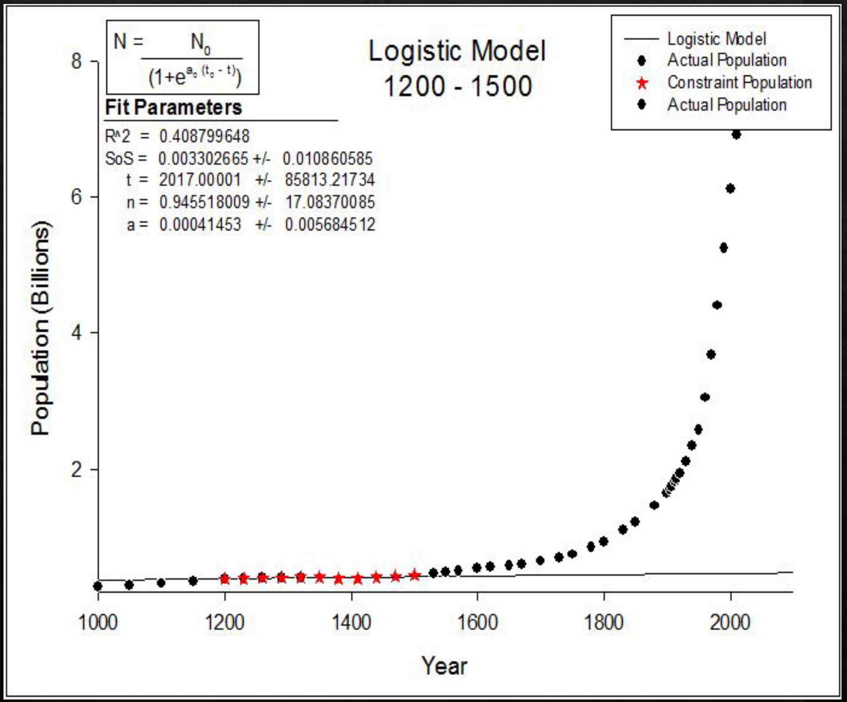


Logistic Model from 1200 – 1300 Vs. 1200 – 1500 Constraint Models

1200- 1300

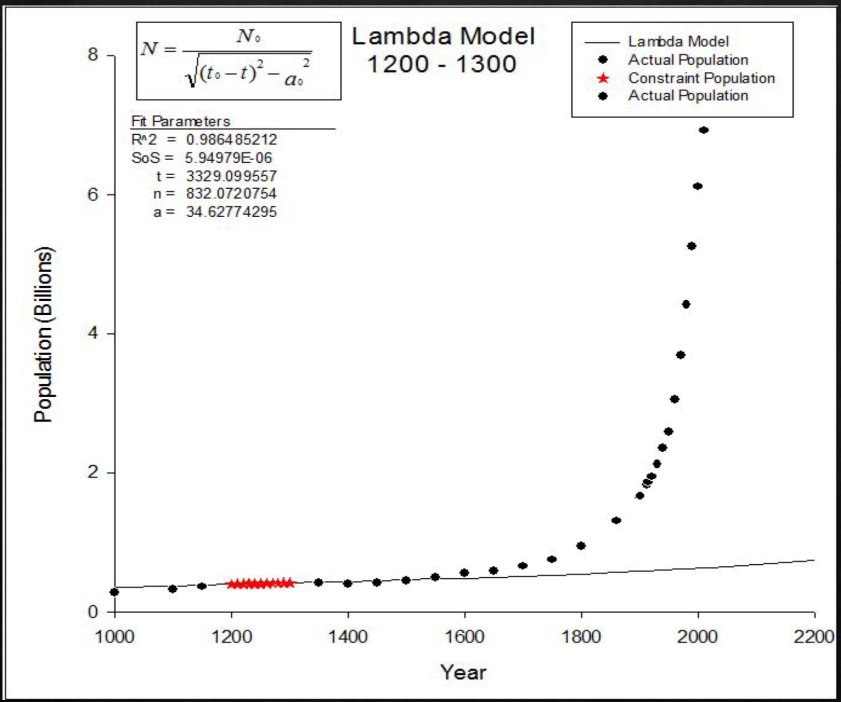


1200 - 1500

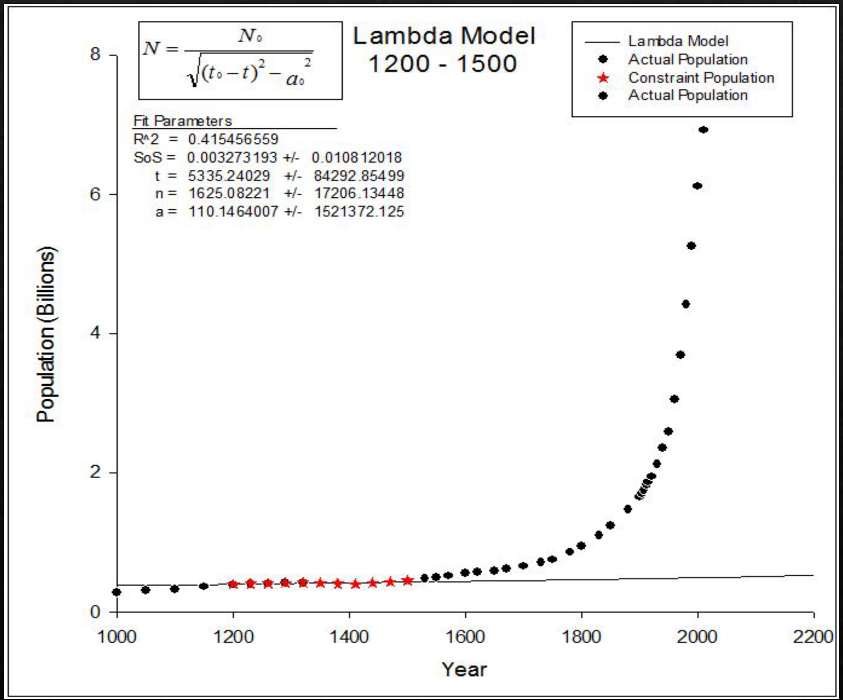


Lambda Model from 1200 – 1300 Vs. 1200 – 1500 Constraint Models

1200 – 1300

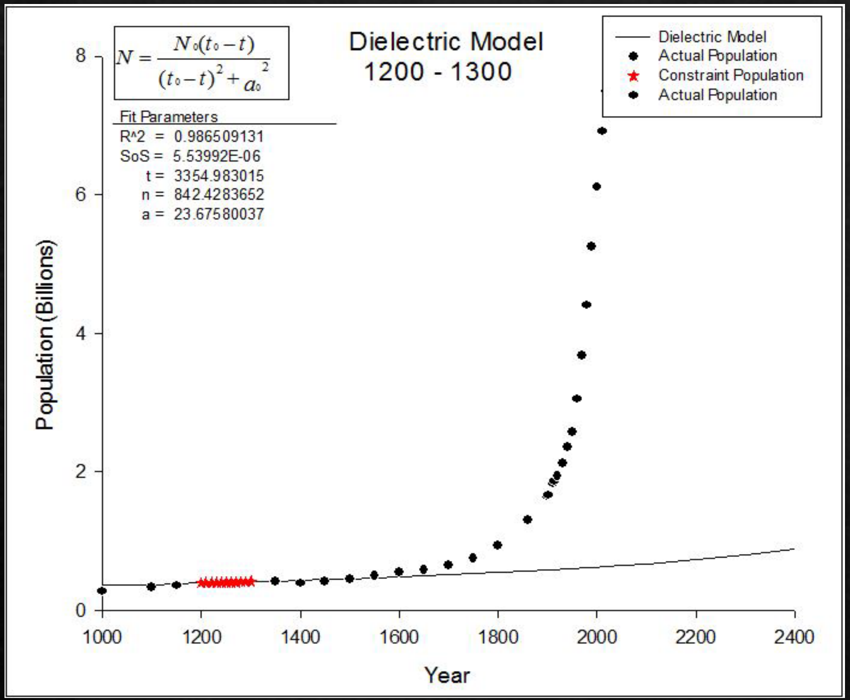


1200 - 1500

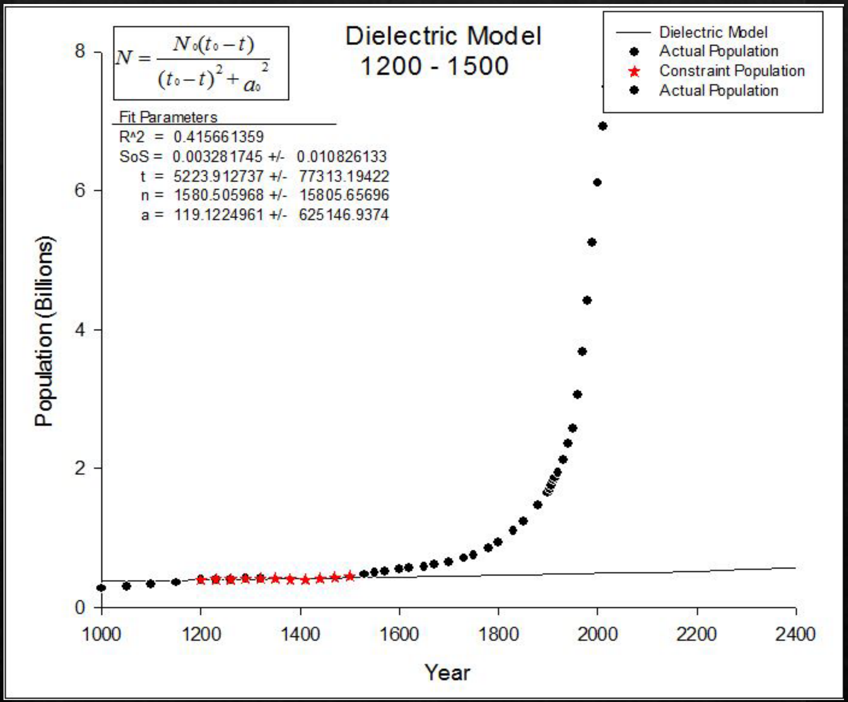


Dielectric Model from 1200 – 1300 Vs. 1200 – 1500 Constraint Models

1200 - 1300

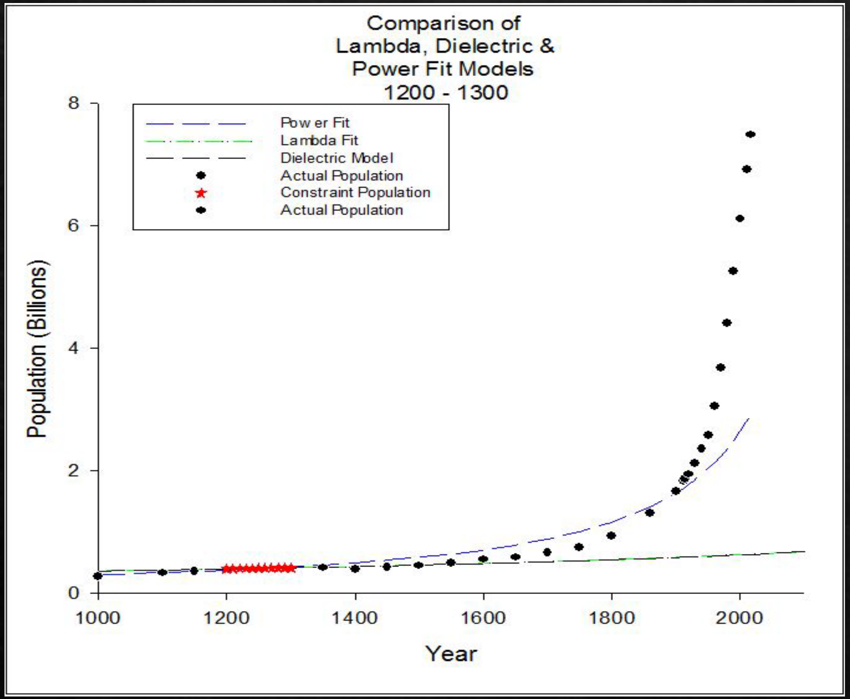


1200 - 1500

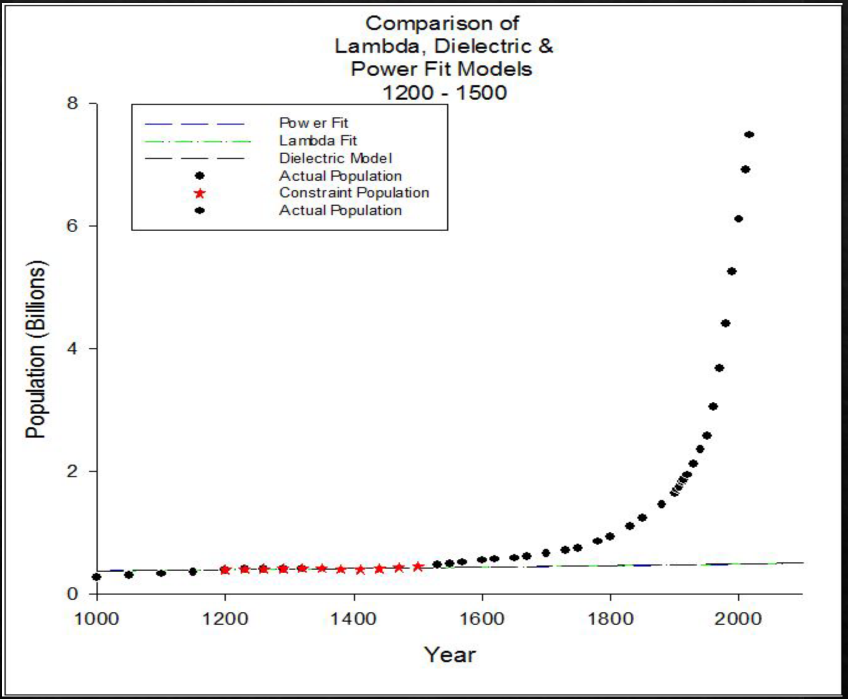


Comparison of the 3 best models from 1200 – 1300 Vs. 1200 – 1500 Constraint Models

1200 - 1300



1200 – 1500



Results of

Curve Fit Parameters		1900 - 1950		1950 - 2017	1650 - 1850	1200 - 1300	1200 - 1500
		Fickess	Prather	Fickess	Fickess	Prather	Prather
Power	R-squared	0.982164345	0.99891634	0.98267513	0.99694839	0.98333879	0.4149905
	SoS	0.089402701	0.00435105	2.6691963	0.00267624	0.00139338	0.00326833
	tt	35599.46569	2083.2552	52120.9082	459.740188	2220.68395	5677.04497
	n	1.846210288	2945.35705	1.916379	1.13087939	2624.60807	2386.19758
	a/tau	2123.82857	1.43643114	2114.91916	2039.79772	1.27913002	1.03609049
Exponential	R-squared	0.977318176	0.99764438	0.98313358	0.99883695	0.98746164	0.41148331
	SoS	0.099760737	0.00947458	2.78534221	2.47463142	4.7143E-06	0.00328771
	tt	5.902966158	2017.00001	10.7535371	5.90296616	2017.00001	2017.00001
	n	0.009389757	4.79719318	0.01560865	0.00938976	0.5690656	0.47759887
	a/tau	2038.036407	0.00922119	2037.70665	2038.03641	0.00045895	0.00023708
Logistic	R-squared	0.977301484	0.99582793	0.98314022	0.97912018	0.98793799	0.40879965
	SoS	0.099808405	0.0167885	2.83331595	0.0173935	4.5351E-06	0.00330266
	tt	1463.199792	2017.00001	1416.48343	119.437623	2017.00001	2017.00001
	n	0.009403647	8.42518064	0.01566636	0.00416367	1.09696464	0.94551801
	a/tau	2623.974362	0.01226355	2348.83072	2958.28227	0.00072275	0.00041453
Lambda	R-squared	0.982839427	0.99883225	0.98282686	0.99771876	0.98648521	0.41545656
	SoS	0.088269344	0.00470153	0.46042124	0.00191277	5.9498E-06	0.00327319
	tt	186.7982124	2017.00001	202.396946	197.040663	3329.09956	5335.24029
	n	47.34639327	205.501564	27.9142348	0.001155	832.072075	1625.08221
	a/tau	2003.866883	41.0650367	2017.45273	2010.99159	34.627743	110.146401
Dielectric	R-squared	0.982895145	0.99850575	0.98232245	0.99771925	0.98650913	0.41566136
	SoS	0.088088758	0.0060085	12.9550099	0.00191278	5.5399E-06	0.00328174
	tt	190.6362732	2032.06737	337.854543	197.03419	3354.98302	5223.91274
	n	29.28641366	222.17166	6.25E-08	0.00151	842.428365	1580.50597
	a/tau	2008.612683	14.6696263	2058.61153	2010.98287	23.6758004	119.122496

Timeframe Extrapolation Conclusion

- This confirms that this method isn't a good way to test the curve fitting optimization.
  - For the most recent time frames there is a good trend in the population which makes that set fit like the total populations set optimization.
  - For the other sets like the 1200 - 1300 the population growth is either too small or inaccurate give this unexpected parameterization model.

New Power Model

- Rather than assuming,

$$\frac{dN}{dt} = aN^2$$

- Let the exponent be given by the variable γ, and the constant by α,

$$\frac{dN}{dt} = \alpha N^{\gamma}$$

- Which, if we let,

$$\gamma = 1 - \frac{1}{\beta}$$

- Yields the solution,

$$N = \left[\frac{\alpha}{\beta}(t - t_0)\right]^{\beta}$$

Comparison to Original Power Model

- To check this model with the original, we let $\gamma = 2$, so $\beta =$ -1,

$$N = \left[\frac{\alpha}{\beta}(t - t_0)\right]^{\beta}$$

$$N = \left[\frac{\alpha}{-1}(t - t_0)\right]^{-1}$$

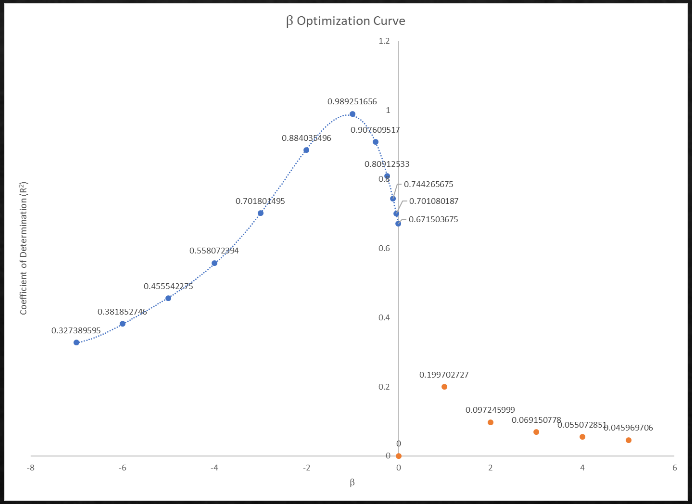
$$N = \frac{1}{\alpha(t_0 - t)}$$

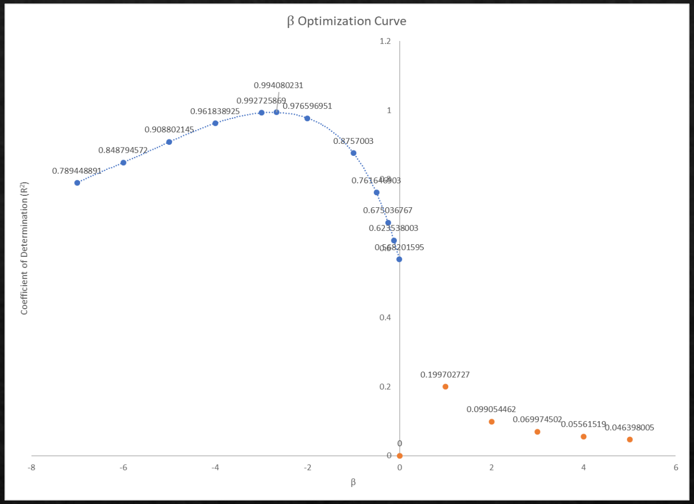
- Which is the result we derived previously, only, we assumed that the exponent could be something other than 1. It cannot be if

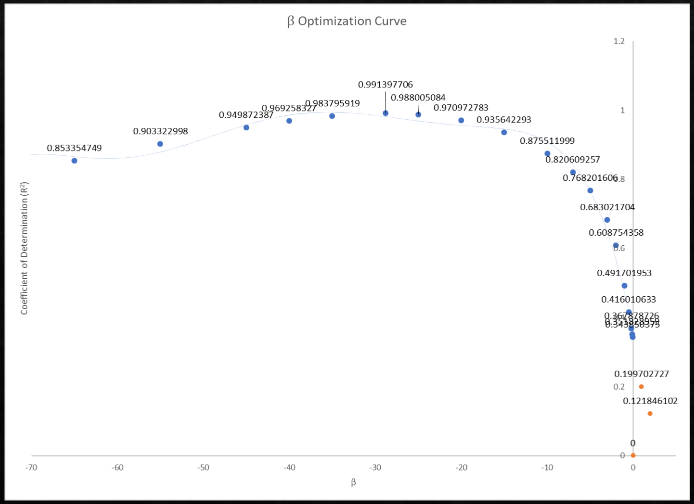
$$\frac{dN}{dt} = aN^2$$

- is confined to have $\gamma = 2$.

Beta Optimization







Optimization Results for Beta Model

- For the optimal values in both the differential form and the functional form are:
  - $\beta = -28.8206$
  - $\gamma = 1.0347$
- Comparing the data itself, a surprising yet expected result occurs:
  - Their fits are nearly identical!

	New Power Model	
	FPM	DPM
R-squared	0.9940802	0.9896781
SoS	17.718725	1806.4124
t_0	2157.0806	
a	0.0087457	0.03225
b	-2.6691052	-2.6691052
c	1.3746574	1.3746574
R-squared	0.9913977	0.9944144
SoS	29.622699	7.5599037
t_0	3829.5994	
a	0.0148026	0.0148026
b	-28.820583	-28.820583
c	1.0346974	1.0346974

Creating Population Model Using Law of Mass Action

Chris

Model of the Reproductive Cycle

Reproduction Cycle

$$M + F \underset{k2}{\overset{k1}{\rightleftarrows}} M + F^{\Xi} \xrightarrow{k3} M + F + C$$

Rate of Children to Adults

$$C \xrightarrow{k4} M$$

$$C \xrightarrow{k5} F$$

Death Rate

$$C \xrightarrow{k6} D$$

$$M \xrightarrow{k7} D$$

$$F \xrightarrow{k8} D$$

Gay or Transgender

$$M \xrightarrow{k9} TW$$

$$W \xrightarrow{k10} \mathrm{T}\Psi$$

Sterile Population

$$M \xrightarrow{k11} \Psi$$

$$F \xrightarrow{k12} \mathrm{W}$$

Properties of the reactions

Constants	Properties
k1	Percent of Pregnancies
k2	Abortions
k3	Birth Rates
k4	Children turn into Man
k5	Children turn into Woman
k6	Children Death Rate
k7	Male Death Rate
k8	Women Death Rate
k9	Men becoming Gay
k10	Women Becoming Gay
k11	Sterile Male
k12	Sterile Women

- To fully model these reactions would be next to impossible.
- The reason for this is that each of these constants would be a function that would also change with time.
- For simplicity in this reaction we will treat these like constants.

Modeling Differential Equations

The differential equation for the total population will be

$$\frac{dP}{dt} = \frac{dF^{\Xi}}{dt} + \frac{dF}{dt} + \frac{dC}{dt} + \frac{dM}{dt} - \frac{dD}{dt} + \frac{dW}{dt} + \frac{d\Psi}{dt} + \frac{d(TW)}{dt} + \frac{d(T\Psi)}{dt}$$

Reproductive Cycle Differentials

- $\frac{dF^{\Xi}}{dt} = k1 * M * F - k2 * F^{\Xi} - k3 * F^{\Xi}$
- $\frac{dF}{dt} = k3 * F^{\Xi} + k2 * F^{\Xi} - k1 * M * F$
- $\frac{dC}{dt} = k1 * M * F$

Children Turning to Adults Differentials

- $\frac{dM}{dt} = k4 * C$
- $\frac{dF}{dt} = k5 * C$

Death Rate Differential

- $\frac{dD}{dt} = k6 * C + k7 * M + k8 * F$

Gay Population Differentials

- $\frac{d(TW)}{dt} = k9 * M$
- $\frac{d(T\Psi)}{dt} = k10 * F$

Sterile Population Differentials

- $\frac{dW}{dt} = k11 * F$
- $\frac{d\Psi}{dt} = k12 * M$

Reproductive Differential Equation

Combining this entire set of reactions together, we get the total population reaction to be:

$$\frac{dP}{dt} = \frac{dF^{\Xi}}{dt} + \frac{dF}{dt} + \frac{dC}{dt} + \frac{dM}{dt} - \frac{dD}{dt} + \frac{dW}{dt} + \frac{d\Psi}{dt} + \frac{d(TW)}{dt} + \frac{d(T\Psi)}{dt}$$

$$\frac{dP}{dt} = [(k1 * M * F - k2 * F^{\Xi} - k3 * F^{\Xi})]$$

$$+[(k3 * F^{\Xi} + k2 * F^{\Xi} - k1 * M * F)]$$

$$+[(k1 * M * F)]$$

$$+[(k4 * C)]$$

$$+[(k5 * C)/$$

$$-[(k6 * C + k7 * M + k8 * F)]$$

$$+[(k10 * F)] + [(k9 * M)]$$

$$+[(k12 *)]M + [(k11 * F)]$$

Future Work & Summary

- Future work
  - Find a way to model the new differential population model, either by looking at the total population or by evaluating sections of the differential model, like the function for pregnant women.
  - Find other differential population models of the other previous models.
  - Do a more thorough investigation of the constraint populations.
- Summary
  - Collect population data
  - Optimize models
  - Make forecasts
  - Compare result
  - Constraint population extrapolation
  - Investigate differential forms

Citations

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- Data Sources:

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