

## The Work-Energy Principle

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Let's start with the classic conservation of energy statement.

$$KE_i + PE_i + W_{nc} = KE_f + PE_f$$

Rearranging to isolate the non-conservative work that is added/extracted from the system.

$$\Delta KE + \Delta PE = W_{nc}$$

Which also says that when there is no non-conservative work, all energy is conserved (stays in the system) and the total energy remains constant.

$$\Delta KE + \Delta PE = 0$$

This is the ideal case where the potential energy is fully converted into useful work. Since useful work is related to motion via the work-energy principle,

$$\Delta KE = W$$

And work performed by a force acting through a distance is given by,

$$W = \int \vec{F} \cdot d\vec{s}$$

So, let's define useful work as  $W$ , exhausted work as  $W_{nc}$ , and natural (stored) work  $W_c$ , which correspond to kinetic energy, dissipative/additive energy, and potential energy, respectively.

$$W = \int \vec{F} \cdot d\vec{s} = \Delta KE$$

$$W_{nc} = \int \vec{F}_{nc} \cdot d\vec{s}$$

$$W_c = \int \vec{F}_c \cdot d\vec{s} = -PE$$

Then, by the conservation of energy,

$$\Delta KE + \Delta PE = W_{nc}$$

$$\int \vec{F} \cdot d\vec{s} - \int \vec{F}_c \cdot d\vec{s} = \int \vec{F}_{nc} \cdot d\vec{s}$$

The energy conservation equation translates to an equation that describes an object's path as it is subject to applied and natural forces.

Let's look at it another way before doing some examples. Consider rearranging to specifically describe the properties that cause changes in motion.

$$\Delta KE = W_{nc} - \Delta PE$$

$$\int \vec{F} \cdot d\vec{s} = \int \vec{F}_c \cdot d\vec{s} + \int \vec{F}_{nc} \cdot d\vec{s}$$

This equation now describes the change in motion of the system in terms of the income and expense (to use financial terms).

$$\int \vec{F} \cdot d\vec{s} = \int \vec{F}_c \cdot d\vec{s} + \int \vec{F}_{nc} \cdot d\vec{s}$$

To break it down:

$W = \Delta KE = \int \vec{F} \cdot d\vec{s}$  – This term is the **net work**. It represents the change in motion of the system resulting from useful work done by a net force acting through a displacement. In practice, it is most usefully expressed as  $\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$  (two measurable speeds) rather than evaluated as a path integral, which would require knowing every force at every point along the trajectory.

$W_c = \Delta PE = - \int \vec{F}_c \cdot d\vec{s}$  – This term is the **conservative work**. It represents the natural source of motion of the system such as a work done by a gravitational field, an electric field, or a spring governed by Hooke's Law. This “work” comes in the form of “stored” potential energy and carries the negative sign to suit. Because conservative forces are **path-independent**, this term reduces to a scalar difference between endpoints:  $mgh$ ,  $q\Delta V$ ,  $\frac{1}{2}kx^2$ . No knowledge of the trajectory is required.

$W_{nc} = \int \vec{F}_{nc} \cdot d\vec{s}$  – This term is the **nonconservative work**. It represents the energy added or removed from the system and is **path-dependent** – different trajectories between the same endpoints yield different results. Dissipative forces such as friction, drag, heat loss, sound, and vibration oppose motion and carry a negative sign. Driving forces such as engines, motors, rockets, or an external push encourage motion and carry a positive sign. This is the only

term that genuinely requires evaluation as a path integral, or more commonly, is inferred by subtraction from the other two.

Since the kinetic and potential energy terms both reduce to endpoint quantities, the energy equation becomes a powerful accounting tool. If the nonconservative work is zero, all stored energy converts without loss into kinetic energy. If energy is added to the system, the change in motion increases. If the system “does work” and energy is lost or extracted, the change in motion decreases.

It is unlikely that all three terms are explicitly known in a given problem. Often, the outcome of some change is of interest – perhaps the final speed of an object, or how far it went, or how long it took to get there. In many cases, the conservative forces are known or easily found and the speeds are measurable, so the nonconservative work – however complex – is determined by the difference. If the problem permits, dissipative forces are included, and the outcome is determined based on the initial and environmental conditions.

## Mechanics

Consider a block of mass  $m$  sliding down a rough plane inclined at  $\theta$  degree and a kinetic friction coefficient  $\mu_k$ . Here, the force of gravity acts uniformly, but at an angle. Friction acts opposite the direction of motion.

$$\Delta KE = W_{\text{nc}} - \Delta PE$$

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = \int \vec{F}_{\text{friction}} \cdot d\vec{s} - (mgh_f - mgh_i)$$

Since friction is defined as,  $F_{\text{friction}} = \mu_k N$ , and the normal force from the inclined plane is described by  $N = mg \cos \theta$ , then,

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = \int_0^L \mu_k mg \cos \theta (-\hat{s}) \cdot d\vec{s} + mgh$$

If the block travels down the inclined plane by a distance  $L$ , and  $h = L \sin \theta$ , then

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = -\mu_k mg \cos \theta L + mgL \sin \theta$$

Which has a clean analytical form to determine the final speed,

$$\frac{1}{2}mv_f^2 = \frac{1}{2}mv_i^2 + mgL(\sin \theta - \mu_k \cos \theta)$$

$$v_f = \sqrt{v_i^2 + 2gL(\sin\theta - \mu_k \cos\theta)}$$

Note: Since  $h = L \sin\theta$ , the quantity  $gL \sin\theta$  is just  $gh$  – the factor  $\sin\theta$  projects slope distance onto the vertical. The units of  $v^2$  match those of gravitational potential  $V_g = gh$ , foreshadowing the role of potential in electric fields.

Also note that if the plane is inclined at  $90^\circ$ , the block is effectively in free-fall and friction is unaffacting (no energy lost to friction), and the length of the plane becomes the height of the “drop”, then the familiar free-fall velocity is recovered:

$$v_f = \sqrt{v_i^2 + 2gh}$$

## Drag Forces

But what if drag forces are considered? What about the noise it makes as it falls? We’ll tackle the first one – I’ll leave you to consider the latter.

Now considering drag forces:

$$\Delta KE = W_{nc} - \Delta PE$$

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = \int \vec{F}_{\text{drag}} \cdot d\vec{s} - (mgh_f - mgh_i)$$

If the drag force is well-described by Stoke’s drag force,  $F_D = 6\pi r\eta v$ , which always opposes the direction of motion, then,

$$\int \vec{F}_{\text{drag}} \cdot d\vec{s} = \int 6\pi r\eta \vec{v} \cdot d\vec{s} = 6\pi r\eta \int \vec{v}(s) \cdot d\vec{s}$$

Since the object’s velocity changes as it falls, we need to integrate velocity as a function of displacement. This can be tedious and highlights the benefit of framing problems in terms of energy rather than force.

For now, let’s determine the velocity of a function of position so that we can evaluate the work done by the drag force over a given displacement.

$$\vec{F} = m\vec{a}$$

$$6\pi r\eta\vec{v} - m\vec{g} = -m\vec{a}$$

$$\vec{v} = \frac{m}{6\pi r\eta}(\vec{g} - \vec{a})$$

The acceleration is not known (because we don't know how the velocity changes - that is precisely question we are trying to solve), so the velocity is not easily tractable.

Rewriting as a first order ordinary differential equation,

$$m \frac{d\vec{v}}{dt} = m\vec{g} - 6\pi r\eta\vec{v}$$

This describes the changing velocity with respect to time. We need to relate how the velocity changes with respect to space (displacement) to how it changes with respect to time.

$$v = \frac{ds}{dt} \rightarrow ds = v dt$$

So,

$$6\pi r\eta \int \vec{v}(s) \cdot d\vec{s} = 6\pi r\eta \int \vec{v}(t) \cdot \vec{v}(t) dt = 6\pi r\eta \int v^2(t) dt$$

This still requires an unknown expression for velocity, but now as a function of time. To determine the velocity requires solving the first-order linear differential equation:

$$m \frac{d\vec{v}}{dt} = m\vec{g} - 6\pi r\eta\vec{v}$$

Whose solution describes the decaying exponential convergence to terminal velocity

$$v(t) = \frac{mg}{6\pi r\eta} (1 - e^{-6\pi r\eta t/m}) + v_i e^{-6\pi r\eta t/m}$$

Which would then be plugged into the nonconservative work integral.

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = mgh - 6\pi r\eta \int_0^t v^2(t) dt$$

Of course, squaring the velocity and then integrating wrt time is menacingly tedious. Inevitably, the nonconservative work integral becomes:

$$W_{\text{drag}} = -6\pi r\eta \int_0^t v^2(t) dt$$

$$= -\frac{m^2 g^2}{6\pi r\eta} t - \frac{m^2 g}{3\pi r\eta} \left( v_i - \frac{mg}{6\pi r\eta} \right) (1 - e^{-6\pi r\eta t/m}) - \frac{m}{2} \left( v_i - \frac{mg}{6\pi r\eta} \right)^2 (1 - e^{-12\pi r\eta t/m})$$

Thus, the entire energy expression becomes,

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = -\frac{m^2 g^2}{6\pi r\eta} t - \frac{m^2 g}{3\pi r\eta} \left( v_i - \frac{mg}{6\pi r\eta} \right) (1 - e^{-6\pi r\eta t/m}) - \frac{m}{2} \left( v_i - \frac{mg}{6\pi r\eta} \right)^2 (1 - e^{-12\pi r\eta t/m}) + mgh$$

As one can see, determining the final speed as a function of the nonconservative forces/work can be quite challenging and the result is quite complex. This solution also requires knowing/measuring the time over which the system evolves (though practically this is usually easy to measure).

### The Energy Audit

Instead, a more common scenario is that the initial and final speeds are known (could be measured) along with the conservative forces and over which distances they act, and one can “audit” the difference to determine how much energy was lost, dissipated, or gained.

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = \int \vec{F}_{\text{drag}} \cdot d\vec{s} - (mgh_f - mgh_i)$$

Since, here  $W_{\text{nc}} = \int \vec{F}_{\text{drag}} \cdot d\vec{s}$ , the above equation can be rearranged as:

$$W_{\text{nc}} = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 - mgh$$

Any result other than zero means there was energy added or removed from the system.

## Electrostatics

Consider a particle of charge  $q$  and mass  $m$  moving in a uniform electric field.

$$\Delta KE = W_{\text{nc}} - \Delta PE$$

Assuming no energy is lost,  $W_{\text{nc}} = 0$ ,

$$\Delta KE = -\Delta PE$$

Where the potential energy is related to the potential (voltage) by a factor of charge,

$$\Delta PE = q\Delta V = q(V_f - V_i)$$

and the potential is related to the electric field strength and separation distance, so

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = -q\Delta V = qEd$$

$$v_f = \sqrt{v_i^2 - \frac{2q}{m}\Delta V} = \sqrt{v_i^2 + \frac{2q}{m}Ed}$$

Which resembles the velocity of the free-falling mass.  $\sqrt{2gh}$  becomes  $\sqrt{\frac{2q}{m}Ed}$

## Constant Retarding Force

Consider that the charged particle is subjected to a resistive dissipative force, much like the friction on the inclined plane.

$$\Delta KE = W_{\text{nc}} - \Delta PE$$

The constant retarding force,  $F_r$ , does work to remove energy from the system, and acts opposite the direction of displacement.

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = \int F_r(-\hat{s}) \cdot d\vec{s} - q\Delta V$$

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = -F_r d - q\Delta V = -F_r d + qEd$$

$$v_f = \sqrt{v_i^2 - \frac{2q}{m}(\Delta V + F_r d)} = \sqrt{v_i^2 + \frac{2}{m}(qE - F_r)d}$$

A similar structure is found. The conservative source ( $qE$  vs  $mg \sin \theta$ ) competes against the dissipative sink ( $f_r$  vs  $\mu_k mg \cos \theta$ ), and the particle accelerates only if the source exceeds the sink. If  $qE < f_r$ , the particle decelerates – just as the block stalls when  $\tan \theta < \mu_k$ .

### Velocity-Dependent Drag

Now consider a velocity-dependent drag force, like that of moving through a fluid. Here, a charged particle moves through a medium where collisions produce a velocity-dependent drag force  $F_D = -\beta v$ . This describes, for example, an ion drifting through a gas, or – at the microscopic level – a conduction electron in a metal lattice.

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = \int \vec{F}_{\text{drag}} \cdot d\vec{s} + qEd$$

The drag work integral becomes:

$$\int \vec{F}_{\text{drag}} \cdot d\vec{s} = -\beta \int v^2(t) dt$$

which requires solving:

$$m \frac{dv}{dt} = qE - \beta v$$

This is the same first-order linear ODE as the Stoke's drag problem, with  $qE$  replacing  $mg$  and  $\beta$  replacing  $6\pi r\eta$ . The solution is:

$$v(t) = \frac{qE}{\beta} (1 - e^{-t/m}) + v_i e^{-t/m}$$

The terminal velocity is  $v_T = qE/\beta$  – the speed at which the electric driving force exactly balances the resistive drag. This is the *drift velocity* of charge carriers in a conductor, and the basis of the Drude model of electrical conduction. The mathematical structure is identical to the falling sphere reaching terminal velocity under gravity and Stoke's drag.

Squaring and integrating produces the same menacingly tedious expression as before, with  $mg \rightarrow qE$  and  $6\pi r\eta \rightarrow \beta$ :

$$\begin{aligned}
W_{\text{drag}} &= -\beta \int_0^t v^2(t) \, dt \\
&= -\frac{q^2 E^2}{\beta} t - \frac{2mqE}{\beta^2} \left( v_i - \frac{qE}{\beta} \right) (1 - e^{-t/m}) - \frac{m}{2} \left( v_i - \frac{qE}{\beta} \right)^2 (1 - e^{-2\beta t/m})
\end{aligned}$$

Once again, the energy audit sidesteps this entirely:

$$W_{\text{nc}} = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 - qEd$$

Measure the initial and final speeds, know the potential difference, and the nonconservative work is determinable – regardless of the complexity of the dissipative force. The potential difference  $\Delta V$  encodes all the conservative (electric field) work in a single scalar quantity, just as  $gh$  encodes all the gravitational work. This is precisely why electric potential is so central to electrostatics: it transforms vector field problems into scalar bookkeeping.

## Closing Remarks

Each term in the energy equation has a different relationship to what is knowable.

The change in kinetic energy is most naturally expressed as  $\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$ : two speed measurements at two moments. One could, in principle, evaluate it as  $\int \vec{F}_{\text{net}} \cdot d\vec{s}$ , but this would require knowing every force acting on the object as a continuous function of position along the entire path, which is precisely the information you usually lack and are trying to determine. If you already knew all of that, you wouldn't need the energy method.

The change in potential energy is path-independent – the definition of a conservative force. Regardless of the trajectory taken between two points, the result is the same:  $mgh$ ,  $q\Delta V$ ,  $\frac{1}{2}kx^2$ . The path integral  $\int \vec{F}_c \cdot d\vec{s}$  exists as a definition and is useful for *deriving* potentials (it is how we obtain  $\Delta V = -\int \vec{E} \cdot d\vec{s}$  in the first place), but in practice the potential at two endpoints is looked up or computed and subtracted. No path is necessary.

The nonconservative work is the stubborn term. It is path-dependent – by definition. In one dimension the distinction is moot (there is only one path), but in two or three dimensions, different trajectories between the same endpoints yield different energy losses: a longer ramp dissipates more to friction, a spiraling charge loses more to drag than one traveling straight. This term genuinely requires either evaluating the integral  $\int \vec{F}_{\text{nc}} \cdot d\vec{s}$  (which, as we have seen, can be menacingly tedious) or, more commonly, by inferring it via subtraction from the two endpoint quantities that *are* knowable.

Two of the three terms reduce to scalar endpoint quantities. The third is determined by their difference. This is the fundamental power of the energy method: it transforms the question

from “what is every force doing at every point along the path?” to “what changed between the beginning and the end?”

In the mechanics examples, the normal force acts perpendicular to the displacement and does zero work – it constrains the trajectory without adding or removing energy. The electromagnetic analog is the magnetic force,  $\vec{F}_B = q\vec{v} \times \vec{B}$ , which is always perpendicular to the velocity by the definition of the cross product. Therefore  $\vec{F}_B \cdot d\vec{s} = 0$  always. It changes the direction of motion (momentum) but not the speed (kinetic energy). Like the normal force, it steers without powering – and is absent from the energy equation entirely.

The velocity-dependent drag discussed above is not merely academic – it is the same physics underlying ion mobility in mass spectrometers, charged particle thermalization in plasmas, and current flow in conductors. At the macroscopic level, this drag is the microscopic origin of Ohm’s law and electrical resistance.